

COUNTING QUASI-IDEMPOTENT IRREDUCIBLE INTEGRAL MATRICES

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ABSTRACT. We study two questions concerning irreducible matrices. The first problem is a counting problem. We count the number of irreducible integral matrices which are annihilated by $X^2 - nX$, for $n > 0$. The second question that we study is whether the set of irreducible, integral matrices which are annihilated by some polynomial $p \in \mathbb{C}[X]$ is finite or not. We show that this is true for positive integral matrices, false for non-negative integral matrices and conjecture that it might be true for irreducible integral matrices.

1. INTRODUCTION

Counting problems are among the most commonly studied problems in combinatorics and a lot of results in this area give rise to integer sequences which can be found on the online encyclopedia of integer sequences (OIES, [13]). In this paper we are going to study the following question. Given a polynomial $p \in \mathbb{C}[X]$, is the number positive (irreducible) integral matrices A such that $p(A) = 0$ finite? The answer to this question is given by the following theorem

Theorem A. *For any polynomial $p \in \mathbb{C}[x]$, the set of positive integer matrices A such that $p(A) = 0$ is finite.*

Now that we know that this set is finite for any polynomial, a natural next question is, if we can count its elements. For an arbitrary polynomial, this seems to be difficult, however, for the polynomials $f_n = X^2 - nX$, we can count the number of positive integral matrices which are annihilated by f_n . We will count these matrices in two different ways. On the one hand, we will count all such matrices and, on the other hand, all matrices up to an equivalence given by permutation of basis vectors. That is, here we say that two $k \times k$ -matrices A, B are equivalent if there exists a $k \times k$ -permutation matrix P_σ , where $\sigma \in S_k$, such that conjugation of A by P_σ yields B .

The motivation to study these questions comes from higher representation theory, more precisely, the following observation. Let A be a finite dimensional \mathbb{C} -algebra of dimension n , then $F := A \otimes_{\mathbb{C}} A$ is an A - A -bimodule. It acts on the category of A - A -bimodules from the left by taking tensor products over A , i.e. for a A - A -bimodule M , the action of F is given by $F(M) = A \otimes_{\mathbb{C}} A \otimes_A M$. Now, we can observe that

$$F^2 = F \circ F = A \otimes_{\mathbb{C}} A \otimes_A A \otimes_{\mathbb{C}} A \cong (A \otimes_{\mathbb{C}} A)^{\oplus n} = F^{\oplus n}.$$

Thus the action of F is quasi-idempotent. Therefore, on the level of the Grothendieck group, F induces a linear transformation which corresponds to a matrix $[F]$ satisfying $f_n([F]) = 0$. These kind of problems appear, for instance, in [11, 12].

The more general problem is motivated by a technique which is used when trying to understand certain 2-representations of finitary 2-categories, see [10]. The main idea here is that a certain element whose action is given by a non-negative, irreducible integral matrix which has to be annihilated by a certain polynomial. With this information one then tries to find all possible such matrices. In small cases this can be done by hand, but the question that occurs quite naturally is whether or not this always is possible, i.e. if there are always only finitely many such matrices. For more details we refer the reader to [7, 11, 12, 14].

In the next section we introduce most of the necessary notation and preliminary results that we will need throughout the paper. In Section 3 we will prove Theorem A. The consecutive section is then spent on counting all irreducible integral matrices satisfying $X^2 = nX$. Finally, in the last section we discuss generalizations of Theorem A and use the structure theory of positively based algebras to prove a similar result that holds for irreducible, non-negative integral matrices but for a restricted set of polynomials.

2. PRELIMINARIES

2.1. Basic definitions and notation. A real matrix A is called *positive* if all entries of A are positive. A real matrix A is called *non-negative* if all entries of A are non-negative. We say that a real square matrix A is *primitive* if it is non-negative and there exists $k > 0$ such that A^k is positive. If A is non-negative and, for each pair i, j , there exists some k such that $(A^k)_{i,j}$ is positive, then we say that A is *irreducible*. Here by $(B)_{i,j}$ we denote the element in the i -th row and the j -th column of B .

For two $n \times m$ matrices A, B we say that $A \leq B$ provided that $(A)_{i,j} \leq (B)_{i,j}$, for all i, j . We write $A < B$, if $A \leq B$ and $A \neq B$. This defines a partial order on $\text{Mat}_{k,k}(\mathbb{Z}_{\geq 0})$ and allows us to write $A > 0$ to express the fact that A is non-negative, where here 0 is used as a shorthand for the $k \times k$ zero matrix.

For $f \in \mathbb{C}[X]$, we define the sets

$$\begin{aligned} K_f^{>0} &= \{A \in \text{Mat}_{k,k}(\mathbb{Z}_{>0}) : k > 0, f(A) = 0\} \\ K_f^{\geq 0} &= \{A \in \text{Mat}_{k,k}(\mathbb{Z}_{\geq 0}) : A \text{ is irreducible}, k > 0, f(A) = 0\} \end{aligned}$$

to be the sets of all irreducible $k \times k$ positive (resp. non-negative) integral matrices which are annihilated by f . Note that we do not restrict the size of A , in particular A can just be a positive 1×1 -matrix which we identify with the corresponding positive integer.

Due to their appearance in, e.g. [11, 14], quasi-idempotent matrices, i.e. matrices that satisfy $X^2 = nX$ are of special interest. Set $f_n(X) := X^2 - nX$, $K_n^{\geq 0} := K_{f_n}^{\geq 0}$ and $K_n^{>0} := K_{f_n}^{>0}$. Then, for a positive (resp. non-negative irreducible) integral matrix A , satisfying $A^2 = nA$ is equivalent to $A \in K_n^{>0}$ (resp. $A \in K_n^{\geq 0}$).

In Section 4 we are going to count the elements in $K_n^{\geq 0}$ in two different ways. Firstly, we are going to simply count all of them. Secondly, we will count all matrices up to permutation of basis vectors. By this we mean the following. Let $A, B \in \text{Mat}_{k,k}(\mathbb{Z}_{\geq 0})$ and denote by S_k the symmetric group on k elements. Then, to each $\sigma \in S_k$, we assign the (permutation) matrix P_σ which is defined by $P_\sigma e_i = e_{\sigma(i)}$, on the elements of the standard basis $\{e_i\} \subseteq \mathbb{R}^k$. Note that P_σ is an orthogonal matrix, i.e. $P_\sigma^{-1} = P_\sigma^t$. We say that A and B are the same up to permutation of basis vectors, denoted $A \sim B$, if there exists $\sigma \in S_k$ such that $P_\sigma^{-1} A P_\sigma = B$. We

will denote the set of all matrices in $K_n^{\geq 0}$ up to permutation of basis vectors by $\overline{K}_n^{\geq 0} := K_n^{\geq 0} / \sim$.

2.2. Compositions and Partitions. Let $n > 0$. A *composition* μ of n , short $\mu \models n$, with k parts, is an element in $\mathbb{Z}_{>0}^k$ such $\mu_1 + \mu_2 + \cdots + \mu_k = n$. We call $\mu \in \mathbb{Z}_{>0}^k$ a *partition* of n , short $\mu \vdash n$, with k parts, if μ is a composition such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$. From now on, whenever we pick a composition or partition of n we assume it to have k parts, unless explicitly stated otherwise, where k is some fixed number between 1 and n .

For two elements $v, w \in \mathbb{Z}_{>0}^k$, we say that v divides w and write $v|w$ if $v_i|w_i$, for all $1 \leq i \leq k$. If $v = (v_1, \dots, v_k)$ and $\sigma \in S_k$, then

$$\sigma(v) = \sigma v = P_\sigma v = (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}),$$

here the product of elements in $\mathbb{Z}_{>0}^k$ is component-wise.

Let p be a prime number and $\mu \models n$, then $\mu^{(p)} := (p^{e_{p,1}}, p^{e_{p,2}}, \dots, p^{e_{p,k}})$, where each $e_{p,i}$ is defined uniquely by the properties $p^{e_{p,i}}|\mu_i$ and $p^{e_{p,i}+1} \nmid \mu_i$. We call $\mu^{(p)}$ the p -part of μ . Then we can write μ as follows:

$$(1) \quad \mu = \prod_{p \text{ prime}} \mu^{(p)} = \prod_{p \text{ prime}} (p^{e_{p,1}}, p^{e_{p,2}}, \dots, p^{e_{p,k}}),$$

where the product is taken component-wise.

2.3. Noetherianity of $\mathbb{Z}_{\geq 0}^k$. For the proofs of our results we will need the following statement.

Theorem 2.1. *The semigroup $\mathbb{Z}_{\geq 0}^k$ is noetherian, for every $k > 0$, i.e. all of its ideals are finitely generated.*

Proof. We may consider $\alpha \in \mathbb{Z}_{\geq 0}^k$ as the exponent of a monomial $X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_k^{\alpha_k}$ in the polynomial ring $\mathbb{C}[X_1, \dots, X_k]$. With this identification, the result follows immediately from [5, Lemma 12.3, page 149]. \square

Now, we can identify the additive semigroup $\text{Mat}_{k,k}(\mathbb{Z}_{\geq 0})$ of non-negative integral $k \times k$ -matrices with the additive semigroup $\mathbb{Z}_{\geq 0}^{k^2}$ and thus we get that every ideal in this semigroup is finitely generated. This will be needed in Section 3.

2.4. Perron-Frobenius Theorem. The second important theorem which we are going to use at different points throughout this work is the *Perron-Frobenius theorem*, more precisely the following version of it.

Theorem 2.2. *Let $A = (a_{i,j}) \in \text{Mat}_{k,k}(\mathbb{R})$ be a non-negative irreducible matrix. Then the following holds.*

- (1) *A has an eigenvalue $r_A = r > 0$, the so-called Perron-Frobenius eigenvalue, of algebraic multiplicity one and such that $r > |\lambda|$, for any other eigenvalue λ of A.*
- (2) *The Perron-Frobenius eigenvalue r satisfies*

$$\min_i \sum_j a_{ij} \leq r \leq \max_i \sum_j a_{ij}.$$

- (3) *If $0 \leq A < B$, then $r_A \leq r_B$. Moreover, if B is irreducible, then the inequality is strict, that is $r_A < r_B$.*

Proof. A proof of these statements can be found in [2, Chapter XIII, §2]. More precisely, (1) is Theorem 2, (2) is Remark 2 and (3) follows from [1, 2.1.5 & 2.1.10]. \square

3. $K_f^{>0}$ IS FINITE FOR GENERAL POLYNOMIALS.

In this section we prove that the set $K_f^{>0}$ is finite, for any $f \in \mathbb{C}[X]$. Before we can do this, we need some notation and a lemma.

Let $x, y \in \mathbb{Z}_{\geq 0}^r$. We say that $x < y$ provided that $x_i \leq y_i$, for all i , and there exists j such that $x_j < y_j$. The reason for this partial order is the fact that we are going to study ideals I in $\text{Mat}_{k,k}(\mathbb{Z}_{\geq 0}) \simeq \mathbb{Z}_{\geq 0}^{k^2}$. By Theorem 2.1, we know that I is finitely generated, say by some B_1, \dots, B_r . Then, for every $X \in I$, we have $X = \sum_{i=1}^r c_i B_i$. So, to every X we can assign its coefficient vector $c_X = (c_i) \in \mathbb{Z}_{\geq 0}^r$. Now, we want to compare matrices in $X, Y \in I$ and then we get that $c_X < c_Y$ implies $X < Y$, so we can study these coefficient vectors instead.

Lemma 3.1. *Let $M \subseteq \mathbb{Z}^r$ be an infinite set. Then there exists an infinite ascending chain in M with respect to $<$ as defined above.*

Proof. Note that M is countably infinite, so there is an enumeration of $M = \{e^{(n)}\}$, where $e^{(n)} = (e_1^{(n)}, \dots, e_r^{(n)})$. Now, since M is infinite, there exists one component in which $e^{(n)}$ is unbounded. Without loss of generality assume it is the first component. Pick a subsequence n_k such that the sequence in the first component of $e^{(n_k)}$ is strictly increasing, i.e. $e_1^{(n_k)} < e_1^{(n_{k+1})}$, for all k . Then there exists a subsequence (n_{k_l}) which is non-decreasing in the second component, i.e. such that $e_2^{(n_{k_l})} \leq e_2^{(n_{k_{l+1}})}$ for l . Similarly, by taking subsequences of subsequences we get a subsequence (n_p) of n_k such that $e_1^{(n_p)} < e_1^{(n_{p+1})}$ and $e_i^{(n_p)} \leq e_i^{(n_{p+1})}$, for all p and all $2 \leq i \leq r$. This yields that $e^{(n_p)}$ is an infinite ascending chain in M . \square

Now we are ready to prove the main result of this section.

Theorem 3.2. *For any polynomial $f \in \mathbb{C}[x]$, the set $K_f^{>0}$ is finite.*

Proof. For $A \in \text{Mat}_{k,k}(\mathbb{Z}_{\geq 0})$ such that $f(A) = 0$, we denote by μ_A the minimal polynomial of A . By definition, we have that μ_A divides f and thus the set of eigenvalues of A is a subset of the zeros of f . Denote by x_0 the zero of f with the highest absolute value. Then, since A is a positive matrix, we can apply the Perron-Frobenius theorem and get that the unique largest eigenvalue $\lambda_A \leq x_0$. Furthermore, we can apply Theorem 2.2.(2) to obtain

$$k \leq \min_i \sum_j a_{ij} \leq \lambda_A \leq x_0,$$

since all $a_{ij} \geq 1$. Thus k is bounded.

Now fix k and assume that the set Y of matrices $A \in \text{Mat}_{k,k}(\mathbb{Z}_{\geq 0})$ such that $f(A) = 0$ is infinite. Consider the ideal $I \subseteq \text{Mat}_{k,k}(\mathbb{Z}_{\geq 0}) \simeq \mathbb{Z}_{\geq 0}^{k^2}$ generated by Y . We want to use the fact that $\mathbb{Z}_{\geq 0}^{k^2}$ is noetherian to obtain a contradiction and thus prove that Y has to be finite.

From Theorem 2.1 we get that I is finitely generated. Let B_1, \dots, B_r be a set of generators of I . Note that f does not necessarily annihilate any of the B_i . Then

we can express every $A \in Y$ as a linear combination of the B_i , i.e.

$$A = \sum_{i=1}^r c_{A,i} B_i.$$

The set $M = \{c_A\}$ of coefficient vectors is an infinite subset of $\mathbb{Z}_{\geq 0}^r$ and thus Lemma 3.1 yields that there is an infinite ascending chain c_{A_k} in M . On the other hand, we have already seen that this means that this is equivalent to having an infinite ascending chain A_k of matrices in Y , with respect to $<$.

However, by Theorem 2.2 (3), this yields that there is an infinite sequence of different Perron-Frobenius eigenvalues, a contradiction, as all eigenvalues, in particular, have to be zeros of f . \square

4. COUNTING QUASI-IDEMPOTENT MATRICES

A first result that we are going to use a lot is the following.

Proposition 4.1. *Let $M \in K_n^{\geq 0}$, then M has rank 1, i.e. there exist $v, w \in \mathbb{Z}_{>0}^k$ such that $M = vw^t$. Moreover, M has trace n .*

Proof. Let $M \in K_n^{\geq 0}$, then M is an irreducible matrix satisfying $M^2 = nM$, i.e. the only possible eigenvalues of M are 0 and n . By Theorem 2.2, we have that n has to be the Perron-Frobenius eigenvalue, in particular all other eigenvalues have to be zero. The claim follows. \square

Moreover, it is easy to see that, if we have a rank 1 matrix $M = vw^t$ with trace n , then $M^2 - nM = 0$, as $M^2 = vw^t vw^t = v(w^t v)w^t = (w^t v)M$ and $w^t v = \sum w_i v_i = \text{tr}(M) = n$. Observe that this implies, in particular, that $M \in K_n^{\geq 0}$ has to be a positive matrix, i.e. $K_n^{\geq 0} = K_n^{>0}$.

These two results together show that, if we want to count matrices in $K_n^{\geq 0}$, we can restrict our attention to pairs $(v, w) \in \mathbb{Z}_{>0}^k \times \mathbb{Z}_{>0}^k$ such that $\sum_{i=1}^k v_i w_i = n$.

4.1. Enumerating $K_n^{\geq 0}$. Our strategy will be to consider all possible diagonals that a matrix in $K_n^{\geq 0}$ can have, in other words, all compositions of n . For each $\mu \models n$, we will consider its divisors. Note that both v and w from above are by definition divisors of μ . Moreover, it is clear that, if we choose one divisor, say v , then there is a unique $w = (w_i) | \mu$ such that $\sum_{i=1}^k v_i w_i = n$, namely $(w_i) = (\mu_i / v_i)$. In other words, it seems to be enough to count all divisors of μ for all compositions of n .

Observe that, if $n = 4$, then $(2, 2) \vdash 4$ with divisors $(2, 2), (2, 1), (1, 2), (1, 1)$. Now, both the pair $((2, 2), (1, 1))$ and the pair $((1, 1), (2, 2))$ yield the same matrix. So, the next goal is to find the correct equivalence relation on the set of divisors of μ such that each equivalence class corresponds to exactly one matrix and vice-versa.

We say that $v_1, v_2 \in \mathbb{Z}_{>0}^k$ are equivalent, short $v_1 \sim_1 v_2$, provided that there exists a positive integer c such that $cv_1 = v_2$ or $v_1 = cv_2$. Given $n > 0$ and $\mu \models n$, we denote by

$$\begin{aligned} K_\mu^{\geq 0} &= \{A \in K_n^{\geq 0} : \forall i \in \{1, 2, \dots, k\} a_{ii} = \mu_i\}, \\ D(\mu) &= \{v \in \mathbb{Z}_{>0}^k : v | \mu\}, \\ D_1(\mu) &= D(\mu) / \sim_1. \end{aligned}$$

Elements in $D_1(\mu)$ are equivalence classes of divisors of μ and will be denoted by $[v]$, provided that $v|\mu$.

Lemma 4.2. *Let $\mu \models n$ and $A \in K_\mu^{\geq 0}$. Then there exists a pair $(v, w) \in D(\mu)^2$ such that $vw^t = A$. Moreover, for every other pair $(\tilde{v}, \tilde{w}) \in D(\mu)$ such that $\tilde{v}\tilde{w}^t = A$, we have $v \sim_1 \tilde{v}$ and $w \sim_1 \tilde{w}$.*

Proof. Let $A = (a_{ij}) \in K_n^{\geq 0}$. Then $\text{tr}(A) = n$ and A has rank one. This implies that A can be written as

$$A = \begin{pmatrix} c_1 \cdot \vec{a} \\ \vdots \\ c_k \cdot \vec{a} \end{pmatrix},$$

for some row vector \vec{a} and positive integers c_1, \dots, c_k . Now we can divide \vec{a} by the greatest common divisor d of its entries and set $v_i = c_i d$. Setting $v_A = v = (v_i)$ and $(w_A) = w = \vec{a}/d$ yields $A = vw^t$. This choice of v and w is unique up to dividing the v_i by a common divisor $e > 0$ and multiplying all w_i by e . However, the resulting vectors are equivalent with respect to \sim_1 . \square

In the above proof, we constructed a map $\hat{\psi}$ from $K_\mu^{\geq 0}$ to $D_1(\mu) \times D_1(\mu)$ mapping a matrix A to the pair $([v_A], [w_A])$ where $(v_A)_i$ is the greatest common divisor of row i and $(w_A)_i$ the first row of the matrix A divided by $(v_A)_1$. Note that we have seen that all rows are equal after dividing by their respective greatest common divisor. In particular, this means that $(w_A)_i = a_{ii}/(v_A)_i$.

Lemma 4.3. *Let $\mu \models n$ be a composition of n with k parts and $v \in D(\mu)$. Then there exists a unique $w_v \in D(\mu)$ such that $v(w_v)^t \in K_\mu^{\geq 0}$. Moreover, if $v \sim_1 \tilde{v}$, then $v(w_v)^t = \tilde{v}(w_{\tilde{v}})^t$.*

Proof. Let $v = (v_1, \dots, v_k) \in D(\mu)$ and set $w_v = w = (w_i)$ with $w_i = \frac{\mu_i}{v_i}$. Then $w \in D(\mu)$ and, moreover, the diagonal of the matrix vw^t equals μ and it is clear that there exists no other $\tilde{w} \in \mathbb{Z}^k$ such that the diagonal of $v\tilde{w}^t$ equals μ . Furthermore, $(vw^t)^2 = vw^t vw^t = n \cdot vw^t$ and thus $vw^t \in K_\mu^{\geq 0}$.

Now, let $v \sim_1 \tilde{v}$. Without loss of generality we assume that there exists $c > 0$ such that $v = c\tilde{v}$. Then $w_v = \left(\frac{\mu_i}{v_i}\right) = \left(\frac{\mu_i}{c\tilde{v}_i}\right) = \frac{1}{c}w_{\tilde{v}}$. This implies the claim, as $vw_v^t = c\tilde{v}w_v^t = \tilde{v}(cw_v)^t = \tilde{v}w_{\tilde{v}}^t$. \square

Theorem 4.4. *The map*

$$\begin{aligned} \varphi : D_1(\mu) &\rightarrow K_\mu^{\geq 0}, \\ [v] &\mapsto vw_v^t, \end{aligned}$$

is a bijection.

Proof. Lemma 4.3 yields that φ defined as above is well-defined. Now, composing $\hat{\psi}$ with the projection on the first component, we get a map $\psi : K_\mu^{\geq 0} \rightarrow D_1(\mu)$, mapping A to $[v_A]$.

Next, let $A \in K_\mu^{\geq 0}$. Then we get that

$$\varphi(\psi(A)) = \varphi([v_A]) = v_A w_{v_A}^t = A,$$

as $(w_{v_A})_i = \frac{\mu_i}{v_i} = \frac{a_{ii}}{(v_A)_i} = \frac{a_{1i}}{(v_A)_i} = (w_A)_i$.

On the other hand, if $[v] \in D_1(\mu)$, then

$$\psi(\varphi([v])) = \psi(vw_v^t) = [\tilde{v}] = [v].$$

Recall that \tilde{v} has the greatest common divisor of the i -th row of $A = vw_v^t$ as its i -th component. In other words, if $\tilde{v} \neq v$, then there exists a greatest common divisor $d > 1$ of the entries of w_v . But then, by construction, we get that $dv = \tilde{v}$ and thus $\tilde{v} \sim_1 v$. \square

This shows that, instead of counting matrices in $K_\mu^{\geq 0}$, we can count elements in $D_1(\mu)$.

Corollary 4.5.

$$|K_n^{\geq 0}| = \sum_{\mu \models n} |K_\mu^{\geq 0}| = \sum_{\mu \models n} |D_1(\mu)|.$$

4.2. Counting $D_1(\mu)$. Let $\mu \models n$ be a composition of n with k parts. Observe that we may assume that, for every $[v] \in D_1(\mu)$, the greatest common divisor of all v_i is 1 due to the fact that we count up to \sim_1 , i.e. up to scalar multiples.

Using the p -part notation for μ , we can now prove a *multiplicative property* of $D_1(\mu)$.

Lemma 4.6. *Let $\mu \models n$ be a composition of n with k parts. Then*

- (i) $|D_1(\mu)| = \prod_{p \text{ prime}} |D_1(\mu^{(p)})|$;
- (ii) $|D_1(\mu^{(p)})| = \prod_{i=1}^k (e_i + 1) - \prod_{i=1}^k e_i$.

Proof. Observe that every divisor of μ is divisible by some divisor of each of the $\mu^{(p)}$. Moreover, these divisors of $\mu^{(p)}$ can be chosen maximally with respect to this property. Then we see that, picking one divisor d_p , for each $\mu^{(p)}$, and multiplying all such, yields a divisor d of μ and that varying the d_p will lead to a result for d .

The number of divisors of $\mu^{(p)}$ equals the product of the number of divisors of every part. For every part we have $e_i + 1$ divisors and thus the total number of divisors of μ_p is $\prod_{i=1}^k (e_i + 1)$. However, this way we count all scalar multiples which we do not want to count when counting $D_1(\mu^{(p)})$. We thus have to assume that, for every divisor $v = (v_1, \dots, v_k)$, at least one $v_i = 1$. Equivalently, we may subtract all divisors v where every $v_i \geq p$. The number of those is the product of all divisors different from 1 for each component. For every component we have e_i such divisors and thus the total number of all such divisors is $\prod_{i=1}^k e_i$. This proves the claim. \square

As an example we will now list the number of matrices for the first n up to 13 there are. It should be remarked that up to $n = 6$ this has been checked both by hand and a matlab code, the values afterwards are calculated using the same matlab code. The corresponding integral sequence seems to be new, cf. [13].

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$ K_n^{\geq 0} $	1	2	6	15	42	99	262	633	1592	3897	9697	23767	58804

In the following example we give a list of all matrices and the corresponding divisor of their diagonals for $n = 4$.

$$\begin{aligned}
(4) &= (1)(4), & \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} (1, 1, 1), \\
\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} (1, 1), & \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (2, 1, 1), \\
\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} (3, 1), & \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} (1, 1, 1), \\
\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} (1, 1), & \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (1, 2, 1), \\
\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1, 3), & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} (1, 1, 1), \\
\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} (1, 1), & \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (1, 1, 2), \\
\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} (1, 2), & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (1, 1, 1, 1), \\
\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} (2, 1),
\end{aligned}$$

Note that $D_1((2, 2)) = \{(2, 2), (2, 1), (1, 2)\}$. So we see that, as $(1, 1) \sim_1 (2, 2)$, we do not include $(1, 1)$ and, in fact, $(1, 1)^t(2, 2)$ yields the same matrix as $(2, 2)^t(1, 1)$ where the latter is counted.

4.3. Counting up to a permutation of basis vectors. In this section we will turn our attention to elements of $\overline{K}_n^{\geq 0}$; that is positive, integral quasi-idempotent matrices up to permutation of basis vectors. This means that we, instead of compositions of n , only need to consider partitions of n as we may assume that the diagonal of a matrix in $\overline{K}_n^{\geq 0}$ is a partition of n , else we can simply permute the basis to obtain a matrix which satisfies this condition.

Let $\mu \vdash n$ be a partition of n . Then we define an equivalence relation on $D(\mu)$ as follows: we say that $v \sim_2 \tilde{v}$, if there exists $c \in \mathbb{Z}_{>0}$ and $\sigma \in S_k$ such that

$$(\sigma(cv) = \tilde{v} \text{ and } \sigma(w_v) = cw_{\tilde{v}}) \text{ or } (\sigma(v) = c\tilde{v} \text{ and } \sigma(cw_v) = w_{\tilde{v}}).$$

Example 4.7. Let $\mu = (4, 2, 2) \vdash 8$ and consider

$$\begin{aligned}
v_1 &= (2, 2, 1), w_1 = (2, 1, 2), \\
v_2 &= (2, 1, 2), w_2 = (2, 2, 1), \\
v_3 &= (1, 2, 2), w_3 = (4, 1, 1).
\end{aligned}$$

Then $v_i, w_i | \mu$ and $v_i w_i^t \in \overline{K}_8^{\geq 0}$. If we just study the action of S_3 on the v_i , then we see that they all lie in the same orbit. However, it is easy to see that $v_3 w_3^t$ cannot be obtained by permutation of basis vectors from one of the other two matrices, as

some of its entries are 8 and such entries do not exist in the other two matrices. On the other hand, we obtain $v_1 w_1^t$ from $v_2 w_2^t$ by swapping the second and the third basis vector. Thus we want $v_1 \sim_2 v_2$ but $v_1, v_2 \not\sim_2 v_3$. It is easy to see that this is, in fact, the case.

Observe that $v \sim_2 \tilde{v}$ is equivalent to

$$(\sigma(cv) = \tilde{v} \text{ or } \sigma v = c\tilde{v}) \text{ and } \sigma\mu = \mu,$$

which follows immediately as $\tilde{v}v_{\tilde{v}} = \mu$, respectively $vw_v = \mu$.

Analogously to the above, we then define

$$D_2(\mu) = D(\mu)/\sim_2.$$

We can now formulate the following:

Theorem 4.8. *Let $\mu \vdash n$ be a partition of n . Then the map*

$$\begin{aligned} \varphi : D_1(\mu) &\rightarrow K_\mu^{\geq 0}, \\ [v] &\mapsto vw_v^t, \end{aligned}$$

induces a bijection $\tilde{\varphi} : D_2(\mu) \rightarrow \overline{K}_\mu^{\geq 0}$.

Proof. Let $[v], [\tilde{v}] \in D_1(\mu)$ and choose v, \tilde{v} such that the greatest common divisor of both all $(w_v)_i$ and all $(w_{\tilde{v}})_i$ is 1. This implies that, for all $c > 1$, we have that $cv, c\tilde{v} \nmid \mu$.

Now $v \sim_2 \tilde{v}$ implies that there exist $\sigma \in S_k, c \in \mathbb{Z}_{>0}$ such that either $\sigma(cv) = \tilde{v}$ and $\sigma w_v = cw_{\tilde{v}}$ or $\sigma v = c\tilde{v}$ and $\sigma(cw_v) = w_{\tilde{v}}$. Assume the former, the latter case can be proved similarly. Note that, by our choice of v and \tilde{v} , we have that $c = 1$.

$$\tilde{v}w_{\tilde{v}}^t = \sigma v \sigma w_v^t = P_\sigma v (P_\sigma w_v)^t = P_\sigma v w_v^t P_\sigma^t = P_\sigma v w_v^t P_\sigma^{-1},$$

i.e. $\tilde{v}w_{\tilde{v}}^t \sim vw_v^t$.

By Theorem 4.4, we know that the inverse of φ is ψ . Now let $A, B \in K_\mu^{\geq 0}$ and assume that $A \sim B$. Then there exists $\sigma \in S_k$ such that $A = P_\sigma B P_\sigma^{-1}$. Moreover, we have

$$v_A w_A^t = A = P_\sigma B P_\sigma^{-1} = P_\sigma v_B w_B^t P_\sigma^t = P_\sigma v_B (P_\sigma w_B)^t = \sigma v_B \sigma(w_B)^t,$$

i.e. $(v_A, w_A) \sim_2 (v_B, w_B)$. This concludes the proof. \square

Thus we get the following formula for enumeration of matrices in $\overline{K}_n^{\geq 0}$.

Corollary 4.9.

$$|\overline{K}_n^{\geq 0}| = \sum_{\mu \vdash n} |D_2(\mu)|.$$

4.4. Counting $D_2(\mu)$. Let $\mu \vdash n$ be a partition of n with k parts and

$$\prod_{p \text{ prime}} \mu^{(p)}$$

its decomposition as in (1). As before, we have that $\mu^{(p)} = (p^{e_1}, \dots, p^{e_k})$, where each e_i is chosen maximally.

Now we can assign a matrix to μ with k columns where the i -th column contains the exponent of the maximal power of the i -th prime dividing μ_i where we ignore

zero rows. This guarantees that the number of rows is bounded. For example, to $\mu = (12, 6, 4, 1) = (2^2, 2^1, 2^2, 2^0) \cdot (3^1, 3^1, 3^0, 3^0)$ we assign the matrix

$$A_\mu = \begin{pmatrix} 2 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Note that the only permutations in S_k which stabilizes μ are those which swap identical columns. That means that, if we want to count the number of equivalence classes with respect to \sim_2 , then we have to count the number of different classes for each set of identical vectors.

First, we can observe that, for each choice that we make in one of the sets of identical vectors, all choices in the other classes give rise to new equivalence classes in $D_2(\mu)$. Thus we can count each set isolated and then take the product over all such sets.

Now, let V be a set of r identical vectors v . In order to understand the number of different equivalence classes, we first note that we can count the number of choices in each component and then take the product over all components as these choices are independent of each other. In other words, we have to count the number of different choices (up to permutation) for a non-negative sequence of length r where every element is less or equal to v_i . This amounts to counting all non-increasing sequence of length r consisting of integers between v_i and 0. Counting this is the same as counting compositions of r with $v_i + 1$ parts, where 0 is an allowed summand or equivalently the number of $v_i + 1$ -tuples of non-negative integers whose sums it r . This number is given by

$$\binom{r + v_i}{r},$$

see e.g. [3, II.4.5. Lemma].

In order to write down a formula for $D_2(\mu)$, we need some notation first. We assign to μ the matrix A_μ , as described above. Assume that $A_\mu \in \text{Mat}_{l,k}(\mathbb{Z})$. Then, for $v \in \mathbb{Z}^l$, we define c_v as the number of times v appears as a column of A_μ . With this, we can now formulate the following result.

Proposition 4.10. *Let $\mu \vdash n$ be a partition of n with k parts and $A_\mu \in \text{Mat}_{l,k}(\mathbb{Z})$ the corresponding matrix of prime factors. Then we have,*

$$D_2(\mu) = \prod_{v \in \mathbb{Z}^l} \prod_{i=1}^l \binom{c_v + v_i}{v_i}.$$

Note that $c_v = 0$ for all but finitely many $v \in \mathbb{Z}^l$ and then $\binom{c_v + v_i}{v_i} = 1$, in particular, we use that $\binom{0}{0} = 1$.

Example 4.11. The following table shows how many matrices up to permutation of basis vectors there are for $n = 1, \dots, 11$. The counting has been performed by hand independently by two people. Again, this integral sequences seems to be new, cf [13].

n	1	2	3	4	5	6	7	8	9	10	11
$ M_n $	1	2	4	8	16	27	51	83	140	225	369

In the following we give a list of all matrices up to permutation of basis vectors and the corresponding divisor of their diagonals for $n = 4$.

$$\begin{aligned}
(4) &= (1)(4), & \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} (1, 1, 1), \\
\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} (1, 1), & \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (2, 1, 1), \\
\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} (3, 1), & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (1, 1, 1, 1), \\
\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} (1, 1), & \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} (1, 2),
\end{aligned}$$

Note that $D_2((2, 2)) = \{(2, 2), (2, 1)\}$. So we see that, as $(1, 1) \sim_1 (2, 2)$, we do not include $(1, 1)$ and, in fact, $(1, 1)^t(2, 2)$ yields the same matrix as $(2, 2)^t(1, 1)$ where the latter is counted. Moreover, we also disregard $(1, 2)$ here since the permutation $(1\ 2)$ stabilizes $(2, 2)$ and turns $(2, 1)$ into $(1, 2)$. Hence $(2, 1) \sim_2 (1, 2)$.

5. GENERALIZATIONS

One natural question is, if the result in Theorem 3.2 can be generalized by relaxing the positivity condition imposed there. As we can see, the second part of the proof where we show that, for a prescribed size of the matrices, there are only finitely many such matrices annihilated by a given polynomial, still holds for non-negative matrices. However, in the general case of non-negative matrices we cannot anymore bound the size of the matrices in the same way as for positive matrices. The reason for that is that we do not have that each entry is at least 1. On the contrary, for non-negative matrices the statement is wrong. Consider for example the polynomial $X - 1$ which annihilates the identity matrix of any given size. This makes non-negative irreducible and primitive matrices the natural candidates to study. Unfortunately, we have neither been able to bound the size for primitive matrices nor to construct an infinite family of non-negative irreducible (primitive) matrices annihilated by the same polynomial. We conjecture:

Conjecture 5.1. An analogue of Theorem 3.2 for primitive matrices is true.

Conjecture 5.2. An analogue of Theorem 3.2 for irreducible matrices is true.

Clearly, Conjecture 5.2 implies Conjecture 5.1.

What we can prove is the following:

Theorem 5.3. Let $f = X^n - a_{n-1}X^{n-1} - \dots - a_1X$ be such that $a_i \geq 0$, for $1 \leq i \leq n-1$, and $a_1 \neq 0$. Then the set $K_f^{\geq 0}$ is finite.

One of the key ingredients in the proof of this proposition is the fact that we can show that there exists an idempotent element which will allow us to bound the size of matrices which can be annihilated by f .

5.1. Positively based algebras. Let \mathbb{k} be a subfield of \mathbb{C} and A a finite dimensional \mathbb{k} -algebra of dimension n . A k -basis \mathbf{B} of A is called *positive*, if all structure constants with respect to \mathbf{B} are non-negative, i.e., for all $i, j \in \{1, 2, \dots, n\}$, we have

$$a_i \cdot a_j = \sum_{l=1}^n c_{i,j}^l a_l,$$

where all $c_{i,j}^l$ are non-negative real numbers. An algebra together with a fixed positive basis is called *positively based*, see [6] for details.

5.2. An example of a positively based algebra. Let $f = X^n - \sum_{i=1}^{n-1} a_i X^i \in \mathbb{R}[X]$ be such that $a_i \in \mathbb{R}_{\geq 0}$ and $I = (f)$ be the ideal in $\mathbb{R}[X]$ generated by f .

Proposition 5.4. *The basis $\{1, X, \dots, X^{n-1}\}$ for $A = \mathbb{R}[X]/I$ is positive.*

Proof. For $i, j \in \{1, 2, \dots, n-1\}$ such that $i+j < n$, we have $X^i \cdot X^j = X^{i+j}$. If $i+j = n$, then

$$X^n = \sum_{i=1}^{n-1} a_i X^i$$

is a linear combination of basis vectors with non-negative coefficients. In particular, $X \cdot X^j$ is a linear combination of basis vectors with non-negative coefficients, for any basis element X^j . Proceeding by induction, we write $X^i \cdot X^j = X \cdot X^{i+j-1}$, where X^{i+j-1} is a linear combination of basis vectors with non-negative coefficients by inductive assumption. Multiplying by X and using the basis of the induction yields the claim. \square

Moreover, one can define the *two-sided preorder* \leq_J on A by setting $X^k \leq_J X^l$ if and only if there are monomials X^m, X^n such that the coefficient of X^l in the product $X^m X^k X^n$ is positive. With this we can define the equivalence relation \sim_J by setting $X^k \sim_J X^l$ if $X^k \leq_J X^l$ and $X^l \leq_J X^k$. The equivalence classes with respect to J are called *two-sided cells*. A two-sided cell \mathcal{J} is called *idempotent* if there exist $x, y, z \in \mathcal{J}$ such that the coefficient of z in xy is positive. In our case, the assumption $a_1 \neq 0$ implies that there are only two two-sided cells $\mathcal{J}_1 = \{1\}$ and $\mathcal{J}_2 = \{X, X^2, \dots, X^{n-1}\}$ and both are idempotent. For more details about the preorder and cells we refer the reader to [6].

5.3. Proof of Theorem 5.3. Let $f = X^n - a_{n-1}X^{n-1} - \dots - a_1X \in \mathbb{R}[X]$ be as above and M an irreducible non-negative, integral $(m \times m)$ -matrix such that $M \in K_f^{\geq 0}$. In order to prove that $K_f^{\geq 0}$ is finite, it suffices to show that m is bounded. The second part of the proof of Theorem 3.2 yields that for a fixed size there can only be finitely many non-negative matrices annihilated by a given polynomial. So we have to find a bound on the size of M .

Now let $A = \mathbb{R}[X]/(f)$ be the quotient of $\mathbb{R}[X]$ by the ideal generated by f . Then A is a positively based algebra and by [6, Proposition 18], we get that there exists an idempotent element

$$e = e(X) = \sum_{i=1}^{n-1} e_i X^i$$

where all e_i are positive real numbers. Now it is clear that $f(M) = 0$ implies that $e(M)$ also is an idempotent, i.e. $e(M)^2 = e(M)$. Moreover, as M is irreducible, we have that, for each i, j , there exists k_{ij} such that $(M^{k_{ij}})_{ij} > 0$. This, together with the fact that all e_i are positive, shows that $e(M)$ is a positive matrix.

Due to Theorem 2.2, we have that $e(M)$ has maximal eigenvalue 1 with multiplicity one and all other eigenvalues are 0. Thus, $e(M)$ has trace 1. Note that the trace of $e(M)$ is a linear combination of the e_i 's with non-negative integer coefficients the sum of which is at least m . This yields that

$$m \cdot \min e_i \leq \operatorname{tr}(M) = 1,$$

which, in turn, gives us that

$$m \leq \frac{1}{\min e_i}.$$

In particular, m is bounded by a quantity which only depends on f . This concludes the proof.

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